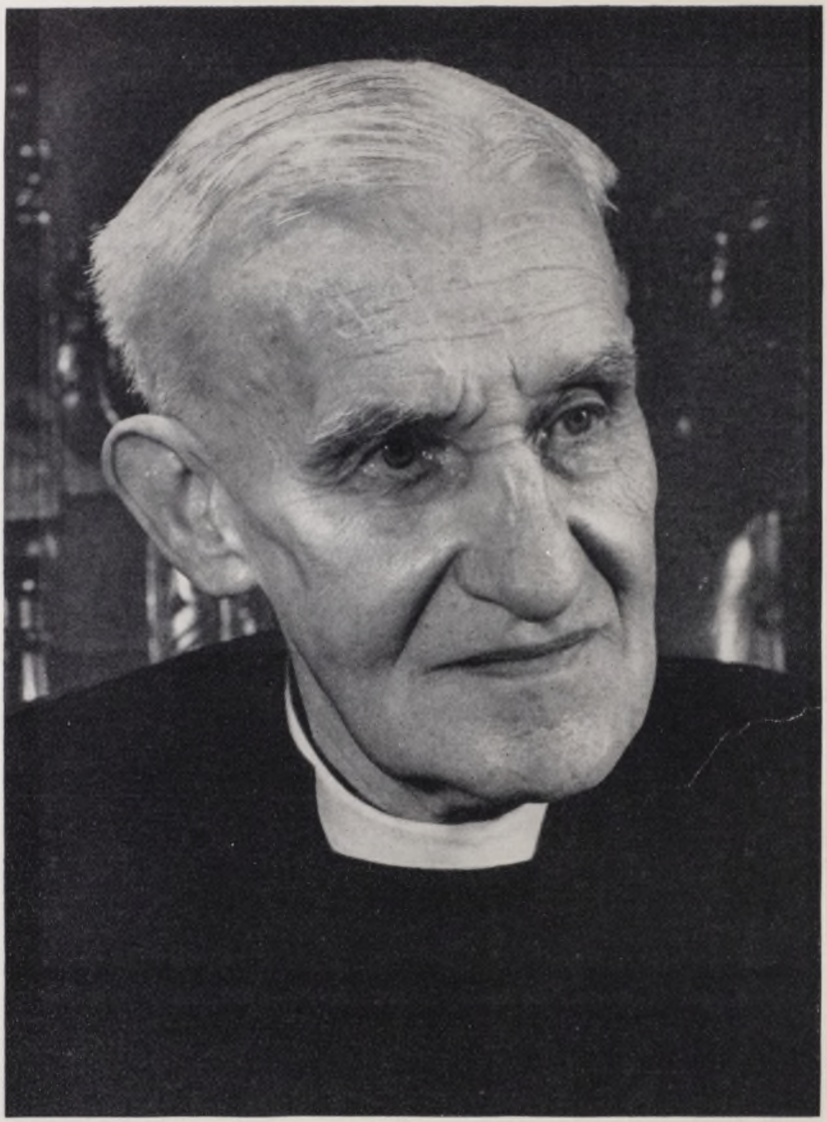

E. W. Birmingham

ERNEST WILLIAM BARNES

1874-1953

The father of Ernest William Barnes was John Starkie Barnes, a native of Accrington in Lancashire, whose forebears and relatives were all engaged in the cotton trade. Mr Barnes became an elementary teacher in the sixties of the last century, and at an early age was appointed a headmaster. His wife, Jane Elizabeth Kerry, who came of an agricultural family in the small Oxfordshire town of Charlbury, was at the time of their marriage headmistress of the associated school for girls. They had a family of four sons, of whom the eldest, the subject of this notice, was born at Birmingham on 1 April 1874: the second, Arthur Stanley (1875—) became M.D., D.Sc., F.R.C.P., and Dean of the Medical Faculty in the University of Birmingham; the third, Alfred Edward (1877-1916) won a classical scholarship at Trinity College, Cambridge, was called to the Bar, and became an official of the Local Government Board; the youngest, James Sidney (1881-1952), was also a scholar of Trinity, was Third Wrangler in the Mathematical Tripos, and entered the Admiralty: he rose to be Deputy Secretary and to be awarded the C.B. and the K.B.E.

Mr J. S. Barnes, after holding more than one headmastership, became Clerk to the King's Norton School Board, and, about 1883, an Inspector of Schools in Birmingham, a position that he occupied throughout the rest of his working life.

Ernest William Barnes was educated first at King Edward's School, Birmingham, where he came under the influence of the headmaster, A. R. Vardy, and the chief mathematics master, Rawdon Levett. In 1892 he won an entrance scholarship in mathematics to Trinity College, Cambridge; he was bracketed Second Wrangler in 1896, was President of the Union in 1897, was placed in the first division of the first class in Part II of the Mathematical Tripos in 1897, and was awarded the first Smith's Prize in 1898. In the latter year he was elected to a Trinity Fellowship, which in 1902 was combined with a college Lectureship and in 1908 with a Tutorship. He graduated Sc.D. of the University in 1907.

Barnes's earliest paper (1)* was a new proof of a theorem due to Picard regarding the integrals of linear differential equations with doubly-periodic coefficients.

In 1899 he wrote a long memoir (2) on the gamma-function, in which he gave the asymptotic expansion of this function at infinity for both real and

* The numbers in brackets refer to the list at the end of this notice.

complex values of the variable, and showed that the gamma-function and Bernoulli's functions can be represented as special cases of the same integral: he proved, moreover, that the gamma-function cannot be the solution of a differential equation whose coefficients are not functions of the same nature. This paper was followed in the same year by (3), in which he studied the G-function, a generalization of the gamma-function which satisfies the equation $G(z+1) = \Gamma(z)\, G(z)$; he showed that it can be expressed as a double product

$$G(z) = e^{r(z)}\, z \prod_{m,\, n\, =\, o}^{\infty}{}' \left\{ \left(1+\frac{z}{m+n}\right) \exp\left[-\frac{z}{m+n} + \frac{z^2}{2(m+n)^2} \right] \right\}$$

where r is a quadratic function of z, and the accent denotes that the pair $m=0$, $n=0$, is to be excluded from the infinite product; he also gave integral expressions for it. He then (4) examined a generalization of the G-function, which had been discovered by Alexejewsky in 1894, and which he named the ***double gamma-function***, since it bears to Weierstrass's sigma-function the same relation as $\Gamma(z)$ bears to sin πz. It satisfies the equation

$$G\left(\frac{z+1}{\tau}\right) = \Gamma\left(\frac{z}{\tau}\right)\; G\left(\frac{z}{\tau}\right)$$

where τ is a constant, and it can be represented by the doubly-infinite product

$$G\left(\frac{z}{\tau}\right) = \exp\left(a\frac{z}{\tau} + b\frac{z^2}{2\tau^2}\right)\frac{z}{\tau}.$$

$$\prod_{m,\, n\, =\, o}^{\infty}{}' \left[\left(1+\frac{z}{m\tau+n}\right) \exp\left\{ -\frac{z}{m\tau+n} + \frac{z^2}{2(m\tau+n)^2} \right\} \right]$$

where a and b are functions of τ only. If a and b are suitably chosen, it satisfies the difference-equations

$$G\left(\frac{z+1}{\tau}\right) = \Gamma\left(\frac{z}{\tau}\right) G\left(\frac{z}{\tau}\right)$$

$$\text{and } G\left(\frac{z+\tau}{\tau}\right) = \left(2\pi\right)^{\frac{1}{2}(\tau-1)}\ \tau^{-z+\frac{1}{2}}\ \Gamma(z)\; G\left(\frac{z}{\tau}\right).$$

The product of four particular functions can be expressed in terms of a Jacobian theta-function. In a long memoir (5) he changed the notation slightly so as to have symmetrical difference-equations involving periods ω_1, ω_2, instead of 1, τ, and worked out many more properties of the double gamma function and related functions. By analogy with the usual definition of the simple Bernoulli function $S_n(a, \omega)$ (whose coefficients involve the Bernoullian numbers) as that polynomial of the $(n+1)$th degree in a which satisfies the difference-equation $f(a+\omega) - f(a) = a^n$ and vanishes with a,

we can define the *double Bernoulli function* ${}_2S_n(a/\omega_1, \omega_2)$ to be that polynomial of the $(n+2)$th degree in a, which vanishes with a and satisfies the difference-equation

$$f(a+\omega_1)-f(a) = S_n(a, \omega_2)-S'_{n+1}(a, \omega_2)/(n+1).$$

This function is symmetric in the parameters ω_1, ω_2. Barnes proved that, apart from a linear and a constant term, ${}_2S_n(a/\omega_1, \omega_2)$ is the only polynomial that satisfies the difference-equation

$$f(a+\omega_1+\omega_2)-f(a+\omega_1)-f(a+\omega_2)+f(a) = a^n.$$

He proved many of its properties, and applied them to established theorems regarding the double gamma function, showing that it can be represented as a singly-infinite product of ordinary gamma-functions, multiplied by quadratic exponential functions of the variable: this form corresponds to the representation of elliptic sigma-functions by infinite products of circular functions. He then discussed its connexion with the Weierstrassian elliptic functions, and with a double Riemann zeta-function defined by

$$\zeta_2\left(s, a/\omega_1, \omega_2\right) = \frac{i\Gamma(1-s)}{2\pi}\int \frac{e^{-az}(-z)^{s-1}\,dz}{(1-e^{\omega_1 z})(1-e^{\omega_2 z})}$$

where the integration is taken along a curve beginning and ending at infinity and enclosing the origin. This function is the simplest solution of the difference-equation

$$f(a+\omega_1+\omega_2)-f(a+\omega_1)-f(a+\omega_2)+f(a) = a^{-s}.$$

Stieltjes had proved that the asymptotic expansion for $\Gamma(x)$ is valid for all values of $|\arg x|$ less than π. Now Barnes, using an idea suggested by Mellin, extended the result to the multiple gamma-functions. The theory of multiple Bernoulli functions, multiple Riemann zeta-functions, and multiple gamma-functions, was studied further in (12).

In (9) he showed that in terms of the double gamma-function it is possible to express the coefficients of capacity of two non-intersecting electrified spheres.

Barnes's next considerable work, (6), was a study of integral functions, a term which he used as a translation of the French *fonctions entières*, i.e. holomorphic transcendental functions: and it was carried further in (8) and (13). He investigated the asymptotic expansions of the logarithms of integral functions of finite order defined by Weierstrassian products, and suggested that such investigations, which characterize the essential singularity at infinity, might lead to a classification of integral functions, relating the classification to Laguerre's introduction of *genre* and Borel's notion of *order*. Barnes's expansions were obtained for functions of simple and multiple linear sequence, these terms being defined as follows: a *simple* integral function is one that may be expressed as a Weierstrassian product whose nth zero a_n depends solely upon n and definite constants, and which is such that the law of dependence of a_n upon n is the same for all but a finite number of

zeros; functions of *multiple linear* sequence are functions whose general zero is of the type $f(\alpha+n_1\,\omega_1+\ldots+n_r\,\omega_r)$, α and the ω's being constants and the n's being the whole numbers that define the particular zero.

The investigation was based on the theory of divergent series: in (6) Barnes developed for this purpose a theory of Borel, who had shown that a power-series with a finite radius of convergence can by means of a certain integral be interpreted for values of the variable outside the circle of convergence. The fundamental procedure consisted in applying the asymptotic expansions of the Euler-Maclaurin sum-formula to a transformation by logarithmic expansion of the function investigated. The terms of the double series which arose in this way were rearranged, and were then summed by an application of Fourier's theorem. Throughout the investigation no attempt was made to determine remainders for the asymptotic expansions. In (8) he further studied the properties of Taylor's expansions of integral functions, and in (13) he examined particularly the asymptotic expansions of functions of multiple linear sequence. As we shall see, the investigations on integral functions defined by Weierstrassian products were continued in (17) and (18).

In a short note (7) published about this time he proved that when $-\pi < \theta < n$, the sum of the Fourier series

$$\sum_{s=-\infty}^{\infty}{}' \frac{(-1)^s e^{si\theta}}{s^{n+1}}$$

can be expressed in terms of the nth Bernoullian function.

In (11) he gave an improved proof of the formula for Euler's constant,

$$\gamma = \int_0^1 \frac{dz}{z}\,(1-e^{-z}-e^{-1/z}).$$

In 1903 and 1905 he discussed the Euler-Maclaurin sum-formula

$$\sum \phi(x) = C + \int^x \phi(x)dx - \tfrac{1}{2}\,\phi(x) + \frac{B_1}{2_1}\,\phi'(x) + \ldots ;$$

in (10) he generalized the analysis connected with it and considered it from the point of view of the theory of asymptotic series: and in (17) he gave a fresh demonstration of the conditions under which his extensions of the formula were valid, and obtained a new form for the remainder.

In (14) he entered a new field, investigating the functional nature of the solution of the homogeneous linear difference-equation of the second order with linear coefficients,

$$(a_2x+b_2)\,f(x+2)+(a_1x+b_1)\,f(x+1)+(a_0x+b_0)\,f(x) = 0.$$

It was shown that in the general cases the solution leads to the complete

system of hypergeometric functions studied by Kummer and Riemann; it may be written

$$f(x) = \tilde{\omega}_1(x) f_1(x) + \tilde{\omega}_2(x) f_2(x)$$

where $\tilde{\omega}_1(x)$ and $\tilde{\omega}_2(x)$ are arbitrary simple periodic functions of x of period 1, while $f_1(x)$ and $f_2(x)$ are one-valued meromorphic functions of x with sequences of poles of similar type. As applications he considered the linear difference-equations for the Legendre and Bessel functions.

In (15) and (16) he studied linear difference-equations of the first order (particular cases of which had occurred in his work on ordinary and double gamma functions)

$$P(x+1) - P(x) = \chi(x).$$

The most simple solution of a linear difference-equation of the first order whose coefficients are meromorphic functions is, in general, a one-valued function with sets of simple sequences of poles tending to infinity. Barnes showed, in connexion with the above difference-equation when $\chi(x)$ is a one-valued analytic function, that, in general, its solution cannot be a solution of any differential equation of finite order unless either: (a) the coefficients of the latter are obtained by differentiation from the solution itself, or (b) we can, from these coefficients and the function $\chi(x)$ and its derivatives, by the fundamental process of forming finite differences coupled with a finite number of elementary algebraic operations, derive the solution itself. The cases of exception to this general theorem were considered. The theorem includes as a special case one proved by Hölder for the gamma function and extended by Barnes in (3) and (5) to the G and double gamma functions. It is important as showing that the linear difference-equation of the first order gives rise to new classes of transcendents which cannot be generated, as are so many functions, by differential equations.

In (16) he investigated in general the nature of the functions defined by linear difference-equations of the first order, considering specially: (i) the existence of a solution, (ii) its analytical expression, (iii) its place among transcendental functions. Let the equation be written

$$\phi(z) f(z+w) - \chi(z) f(z) = \psi(z)$$

where $\phi(z)$, $\chi(z)$, and $\psi(z)$ are assumed to be analytic functions of z.

This may at once be reduced to two others of simple type:

$$\frac{f_1(z+w)}{f_1(z)} = \frac{\phi(z)}{\chi(z)} \tag{A}$$

and

$$f_2(z+w) - f_2(z) = \frac{\psi(z)}{\chi(z)} f_1(z) \tag{B}$$

with

$$f(z) = \frac{f_2(z)}{f_1(z)}.$$

He then regards (A) and (B) as the two fundamental equations, taking $\phi(z)$, $\chi(z)$ and $f(z)$ to be integral functions as studied in (6), (8), and (13).

He considers the solutions of these equations under very general conditions, and finally deduces the nature of the solution of the original equation.

In (18) Barnes returned to the asymptotic expansion of integral functions defined by Weierstrassian products, which he had considered in (6), (8), and (13), but he now approached the subject from a different point of view, avoiding dependence on the theory of divergent series and returning to the methods used in (2) and (5) to obtain the asymptotic expansions of the simple and double gamma functions. The procedure is an application of Pauchy's theory of residues, suggested by an investigation in which Mellin had applied it to the case of the simple gamma-function.

In (19) he continued the study of asymptotic expansions, considering now the asymptotic expansions of functions that are defined by Taylor series instead of Weierstrassian products. The fundamental procedure is based upon the following theorem: suppose that we wish to find an asymptotic expansion when $|x|$ is large for the integral

$$I = \frac{\mathrm{i}}{2\pi} \int_{c} \mathrm{e}^{-xz} f(z)\ (-z)^{\beta-1}\, \mathrm{d}z$$

taken round a contour C (as used in the theory of gamma-functions) which encloses the origin and embraces an axis drawn from the origin to infinity along which $R(xz)$ is positive: $f(z)$ being a function which for values of $|z|$ less than l admits the convergent expansion

$$f(z) = \sum_{n=0}^{\infty} c_n(-z)^n.$$

Then the integral I admits the asymptotic expansion

$$\sum_{m=0}^{\infty} c_m \frac{\mathrm{i}}{2\pi} \int_{C} \mathrm{e}^{-xz}(-z)^{\beta+n-1}\, \mathrm{d}z = \sum_{m=0}^{\infty} \frac{c_n}{\Gamma(1-\beta-n)\ x^{\beta+n}}\,.$$

Barnes considers various standard types of integral functions defined by Taylor series, and for each function finds the behaviour at infinity, applying his new methods of contour integration so as to get complete asymptotic expansions.

(20), (21), (22), and (23) are developments of (19). In (21) he studied specially the case $\beta = 1$ of the function

$$G_\beta(x, \theta) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n+1)\ (n+\theta)^\beta}\,.$$

In (22) he applied the investigation given in (19) specially to the functions defined when $|x| < 1$ by the Taylor series

$$\sum_{n=0}^{\infty} \frac{x^n}{(n+\theta)^\beta}$$

This function has a single singularity in the finite part of the plane, at $x = 1$: it is not an essential singularity, but the function is many-valued in its neighbourhood. He found its asymptotic expansion, and in the second part of the paper obtained analogous theorems for a function which is defined when $|x| < 1$ by an expansion

$$\sum_{n=0}^{\infty} \frac{x^n\, \chi(n+\theta)}{(n+\theta)^{\beta}}$$

where outside a certain circle, $\chi(x)$ admits an absolutely convergent expansion as a series of negative powers of x.

In (24) he obtained the asymptotic expansion of integral functions defined by the generalized hypergeometric series

$$F(x) = 1 + \frac{\alpha_1\, \alpha_2 \ldots \alpha_p}{1.\, \rho_1 \rho_2 \ldots \rho_q} x + \frac{\alpha_1(\alpha_2+1)\ldots\alpha_p(\alpha_p+1)}{1.\,2.\, \rho_1\, (\rho_1+1)\ldots\rho_q\,(\rho_q+1)} x^2 + \ldots \qquad (q \geqslant p)$$

and this gave a simple proof of some theorems found earlier by Orr.

In all Barnes's papers from (25) onwards he uses integrals in which the integrand involves gamma functions of the variable of integration, with a path of integration which is parallel to the imaginary axis, with loops if necessary. Such integrals had been introduced originally by Mellin, but it was from the writings of Barnes that British mathematicians first realized how powerful a tool they were.

In (25) he gave a proof of Abel's theorem on the binomial expansion for a complex variable and complex index, by making use of the integral

$$\frac{1}{2\pi \mathrm{i}} \int_L \Gamma(\phi - s)\ \Gamma(-\phi)\ x^{\phi}\ \mathrm{d}\phi \equiv F(x,\, s)$$

where s and x are complex numbers, and x^{ϕ} is $\exp(\phi \log x)$: L is a path of integration which is parallel to the imaginary axis and cuts the real axis between $\phi = 0$ and $\phi = -1$ (with a loop, if necessary) so that s lies on the left of L. Whatever the value of $|x|$ may be, the integral is convergent for all values of s, if $|\arg x| < \pi$, and, moreover, is convergent for $|\arg x| = \pi$ when $R(s) > 0$. For $s = 0, 1, 2, \ldots$, the integral does not exist.

Barnes shows, that for all values of x and s, for which $|\arg x| < \pi$ and $s \neq 0, 1, 2, \ldots$, we have $F(x, s) = -\Gamma(-s)\ (1+x)^s$, where $(1+x)^s = \exp\{s \log(1+x)\}$ and $|\arg(1+x)| < \pi$, and that

$$(1+x)^s = 1 + \frac{s}{1} x + \frac{s(s-1)}{1.2} x^2 + \ldots$$

whenever the series is convergent.

In (26) he makes use of factorial series of the form

$$\sum_{n=0}^{\infty} \frac{\Gamma(n-s)}{\Gamma(n+1)\ (n+s)^{\beta}}$$

in an asymptotic expansion.

In (27) he gave a new development of the theory of the hypergeometric function, which avoided certain difficulties that are found in the works on this subject by Riemann and Pochhammer. He considers the contour integral

$$2\pi i \int_{-\infty i}^{\infty i} \frac{\Gamma(a+s)\ \Gamma(b+s)\ \Gamma(-s)}{\Gamma(c+s)} (-z)^s\, ds$$

where the path of integration is curved (if necessary) so as to ensure that the poles of $\Gamma(a+s)\ \Gamma(b+s)$, namely, $s = -a-n, -b-n$, $(n = 0, 1, 2, ...)$ lie on the left of the path, and the poles of $\Gamma(-s)$, namely, $s = 0, 1, 2, ...,$ lie on the right of the path. He shows that this integral represents an analytic function so long as $|\arg z| < \pi$, and by the theory of residues it is found that when $|z| < 1$, this function may be represented by the series

$$\sum_{n=0}^{\infty} \frac{\Gamma(a+n)\ \Gamma(b+n)}{\Gamma(c+n)\ n!} z^n.$$

When $|z| > 1$, the integral can be transformed into the sum of two power-series in z^{-1}, and thus Barnes obtained the result

$$\begin{aligned}\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} F(a, b : c : z) = {} & \frac{\Gamma(a)\ \Gamma(a-b)}{\Gamma(a-c)} (-z)^{-a} F(a, 1-c+a; 1-b+a; z^{-1}) \\ & + \frac{\Gamma(b)\ \Gamma(b-a)}{\Gamma(b-c)} (-z)^{-b} F(b, 1-c+b; 1-a+b; z^{-1})\end{aligned}$$

when $F(a, b : c : z)$ denotes the hypergeometric series, and $|\arg(-z)| < \pi$.

In the course of the work the value of the integral

$$\int_{-\infty i}^{\infty i} \Gamma(\alpha_1+s)\ \Gamma(\alpha_2+s)\ \Gamma(\beta_1-s)\ \Gamma(\beta_2-s)\ ds$$

is found in terms of gamma functions; a result often called 'Barnes' lemma'.

In (28) he investigated the Associated Legendre functions $P_n^m(z)$ and $Q_n^m(z)$ with arbitrary indices n, m, i.e. the solutions of the differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left\{n(n+1) - \frac{m^2}{1-x^2}\right\} y = 0$$

for arbitrary values of n and m, proceeding in much the same way as in the

definition of derivatives of arbitrary index. It is first shown, that the integral

$$I_m(x) = -\frac{1}{2\pi i}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m} \int \frac{\Gamma(s-n)\,\Gamma(n+1+s)\,\Gamma(-s)}{\Gamma(1-m+s)}\{\tfrac{1}{2}(x-1)\}^s\,ds$$

satisfies the above equation, and likewise $I_{-m}(x)$. In this the path of integration is parallel to the imaginary axis, with loops when necessary in order to secure that all positive sequences of poles lie on the right of the path of integration, and all negative on the left. By means of this integral the functions P_n^m and Q_n^m with arbitrary complex indices n, m, are defined for all values of x outside a cut drawn from $x = -\infty$ to $x = +1$, and indeed we have

$$P_n^m(x) = -\frac{\sin(n\pi)}{\pi} I_m(x), \quad Q_n^m(x) = \tfrac{1}{2}\{I_m(-x) - e^{\pm n i\pi} I_m(x)\}.$$

A great many formulae, particularly various representations of P_n^m and Q_n^m by means of hypergeometric series, are obtained by taking the integral I_m along a closed path of integration and applying Cauchy's theorem. In the second chapter the asymptotic values of the functions are obtained for $n\to\infty$ and $m\to\infty$, and for various values of x. In the third chapter a generalization of Rodrigues's formula is given, and various recurrence-formulae obtained and definite integrals calculated.

The last work of Barnes's career as a professional mathematician, (29), is a short paper on generalized hypergeometric series. Let

$$F(\alpha_1, \alpha_2, \alpha_3 : \beta_1, \beta_2, 1) = 1 + \frac{\alpha_1\,\alpha_2\,\alpha_3}{\beta_1\,\beta_2\,1} + \frac{\alpha_1(\alpha_1+1)\alpha_2(\alpha_2+1)\alpha_3(\alpha_3+1)}{\beta_1(\beta_1+1)\,\beta_2(\beta_2+1)\,1.2} + \dots$$

the series being convergent when $R(\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3)>0$, and the function being defined to be the analytic continuation of the series when the inequality is not satisfied. Then it had been shown many years before by Thomae, but was now shown much more briefly and elegantly by Barnes, that

$$F(\alpha_1, \alpha_2, \alpha_3 : \beta_1, \beta_2, 1)$$
$$= \frac{\Gamma(\beta_2)\,\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3)}{\Gamma(\beta_2-\alpha_1)\,\Gamma(\beta_1+\beta_2-\alpha_2-\alpha_3)} F(\alpha_1, \beta_1-\alpha_2, \beta_1-\alpha_3 : \beta_1, \beta_1+\beta_2-\alpha_2-\alpha_3, 1).$$

The utility of the result depends on the fact that one of the series may be rapidly convergent while the other converges very slowly.

Barnes was elected a Fellow of the Royal Society in 1909.

His father, Mr J. S. Barnes, was a Baptist, active in Church work: but Ernest, when a boy of about fifteen at King Edward's school, joined the Anglican Church. In 1902 he was ordained, and from 1906 to 1908 was Junior Dean of Trinity. In 1915 he left Cambridge, when, on the nomination of Mr Asquith's government, he was appointed Master of the Temple. In the following year he married Adelaide Caroline Theresa, only daughter of Sir Adolphus W. Ward, Litt.D., F.B.A., Master of Peterhouse. In 1918

he was nominated by the Prime Minister, Mr Lloyd George, to a Canonry of Westminster, and in 1924, by Mr Ramsay MacDonald, to the Bishopric of Birmingham. He was a Select Preacher at both Cambridge and Oxford on many occasions.

His appointment to high office in the Church brought to an end his life as a professional mathematician. In 1933, however, he published a volume of Gifford lectures delivered in the University of Aberdeen, under the title *Scientific theory and religion: the world described by science and its spiritual interpretation.* The science which is expounded in this book is not, however, that branch in which he was an acknowledged master, namely pure mathematics, but a number of different branches, some of which he must have learnt late in life: for two of the most important of them had been discovered only about the time when he exchanged Trinity for London, and much of the information given is of a date more recent than his elevation to the episcopal bench. One cannot but marvel at the ability and industry which had enabled him, in the intervals of his duties as a Bishop, to become acquainted with such an immense field of new knowledge.

He received the honorary degree of D.D. from the Universities of Aberdeen and Edinburgh, and the honorary degree of LL.D. from the University of Glasgow.

In 1952 he retired, on account of ill-health, and on 29 November 1953 he died. He was survived by Mrs Barnes and two sons.

My grateful thanks are due to Mrs Barnes, to Dr Stanley Barnes, and to Professor W. N. Bailey, for much help in the preparation of this Notice.

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